GRONDSLAGEN VAN GELUID EN GELUIDVOORTPLANTING

Diemer de Vries
ex - TU Delft

Doel van de presentatie

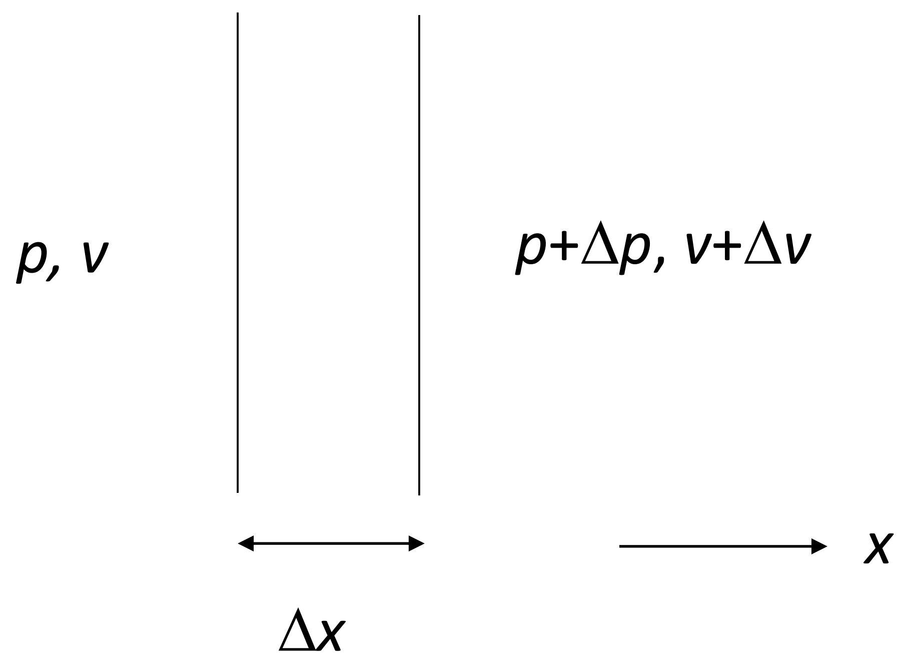
Het nog eens ophalen van de basisprincipes van geluid en geluidvoortplanting, beschreven door *natuurkundige* wetten en daaruit voortvloeiende (relatief simpele) *wiskundige* vergelijkingen

GELUID BESTAAT UIT GOLVEN!

- Kenmerkend voor een golf:
  * het medium blijft per saldo 'op z'n plek'
  * een *verstoring* (verplaatsing + vervorming) beweegt zich door het medium
- Een golf *buigt* om objecten en hoeken, in tegenstelling tot een straal
- Wees voorzichtig met *ray tracing* technieken!

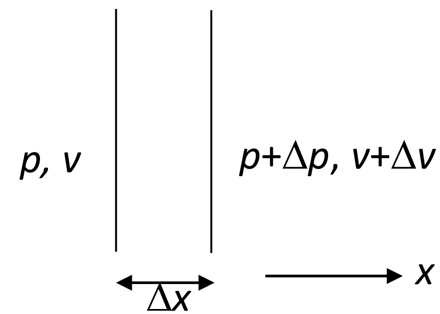
Basics of 1D sound propagation (1)

- System: slice of fluid with mass density ρ and compression modulus K



Basics of 1D sound propagation (2)

Δx is chosen such small that *gradients* of p and v are *constant*:



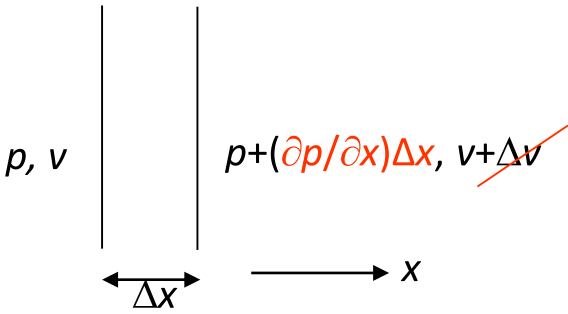
$$\Delta p = \frac{\partial p}{\partial x}\Delta x,\ \Delta v = \frac{\partial v}{\partial x}\Delta x$$

Due to differences in pressure and velocity, the slice is simultaneously *translated* and *compressed*

In the *linear* situation, both phenomena can be treated separately and combined at the end

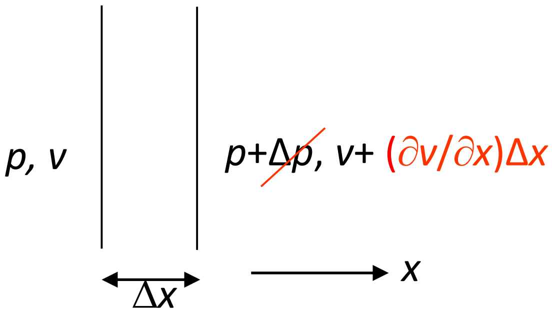
Translation

- Translation is described by *Newton*'s second law: "*F* = *ma*"
- For a surface area unit of the slice:



$$-\frac{\partial p}{\partial x}\Delta x = \rho \Delta x \frac{\partial v}{\partial t}$$

Compression/expansion



- Compression/expansion is described by *Hooke*'s law: "$\sigma = E\varepsilon$"
- For a surface area unit of the slice:

$$-p = K\frac{\Delta(\Delta x)}{\Delta x} = K\frac{\frac{\partial(\Delta x)}{\partial x}\Delta x}{\Delta x} = K\frac{\partial(\Delta x)}{\partial x}$$

or, derived to time:

$$-\frac{\partial p}{\partial t} = K\frac{\partial v}{\partial x}$$

Newton + Hooke = wave equation

Newton: $$-\frac{\partial p}{\partial x} = \rho \frac{\partial v}{\partial t} \rightarrow -K\frac{\partial^2 p}{\partial x^2} = K\rho\frac{\partial^2 v}{\partial x \partial t}$$

Hooke: $$-\frac{\partial p}{\partial t} = K \frac{\partial v}{\partial x} \rightarrow -\rho\frac{\partial^2 p}{\partial t^2} = K\rho\frac{\partial^2 v}{\partial x \partial t}$$

$$c = \sqrt{\frac{K}{\rho}}$$

$$-K\frac{\partial^2 p}{\partial x^2} + \rho\frac{\partial^2 p}{\partial t^2} = 0 \rightarrow$$

$$\frac{\partial^2 p}{\partial x^2} - \frac{\rho}{K}\frac{\partial^2 p}{\partial t^2} =$$

Newton + Hooke = wave equation

$$\text{Newton:}\quad -\frac{\partial p}{\partial x} = \rho\frac{\partial v}{\partial t} \rightarrow -K\frac{\partial^2 p}{\partial x^2} = K\rho\frac{\partial^2 v}{\partial x \partial t}$$

$$\text{Hooke:}\quad -\frac{\partial p}{\partial t} = K\frac{\partial v}{\partial x} \rightarrow -\rho\frac{\partial^2 p}{\partial t^2} = K\rho\frac{\partial^2 v}{\partial x \partial t}$$

$$c = \sqrt{\frac{K}{\rho}}$$

$$-K\frac{\partial^2 p}{\partial x^2} + \rho\frac{\partial^2 p}{\partial t^2} = 0 \rightarrow$$

$$\frac{\partial^2 p}{\partial x^2} - \frac{\rho}{K}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

Why eliminating *v*, not *p* ?

- *p* is scalar → mathematically simpler than vector *v*
- Our ears are *p*-sensitive (this is the real argument)

Solution of the wave equation

- General solution: $p(x,t) = s_1(t - \frac{x}{c}) + s_2(t + \frac{x}{c})$
- Interpretation: *plane* waves in +*x* und –*x* direction with *propagation velocity c*:

p – value $s_1\left(t_0 - \frac{x_0}{c}\right)$ is found back after Δt at

$$x_0 + \Delta x \;\rightarrow\; t_0 - \frac{x_0}{c} = t_0 + \Delta t - \frac{x_0 + \Delta x}{c} \rightarrow$$

$$c = \frac{\Delta x}{\Delta t}$$

Summary:

- Acoustic wave is combination of translation and compression/expansion
- Translation described by Newton: $-\dfrac{\partial p}{\partial x} = \rho \dfrac{\partial v}{\partial t}$
- Compression/expansion described by Hooke: $-\dfrac{\partial v}{\partial x} = \dfrac{1}{K}\dfrac{\partial p}{\partial t}$
- Combination gives wave equation:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0;\ c = \sqrt{\frac{K}{\rho}}\ \text{is propagation velocity}$$

- Solution: plane waves in +/- *x* direction

Vraagje tussendoor:

- Zijn er ook situaties waarbij alleen translatie (massa, Newton) een rol speelt?
- Antwoord: ja, wanneer het medium zich bevindt in een *aan beide zijden open* cylinder met afmetingen *kleiner* dan de golflengte: het medium beweegt als een ‘propje’ (massaatje)

En dus ook:

- Zijn er situaties waarbij alleen compressie/expansie (stijfheid, Hooke) een rol speelt?
- Antwoord: ja, wanneer het medium zich bevindt in een *gesloten volume* met afmetingen *kleiner* dan de golflengte met (een) enkele opening(en) waardoor trillingen kunnen toetreden: het medium ‘kan niet weg’ en wordt als een ‘veer’ vervormd

Helmholtz resonator

- Voeg een ‘akoestische massa’ en een ‘akoestische veer’ samen (vgl. lege fles) en je hebt een zgn. Helmholtz resonator

- Hierover gaat Theo het straks hebben.

Helmholtz equation + solution

- Temporal Fourier Transform of wave equation is called the *Helmholtz* equation
- Remember: $\frac{\partial}{\partial t} \xrightarrow{FT} j\omega \;;\; \frac{\partial^2}{\partial t^2} \xrightarrow{FT} -\omega^2;\; +\Delta t \xrightarrow{FT} \exp(j\omega\Delta t)$
- Convention: $p(t) \xrightarrow{FT} P(\omega)$
- Hence:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

$$\frac{\partial^2 P(x,\omega)}{\partial x^2} + \frac{\omega^2}{c^2} P(x,\omega) = \frac{\partial^2 P(x,\omega)}{\partial x^2} + k^2 P(x,\omega) = 0$$

wave number

- Solution: $P(x,\omega) = S_1(\omega)\exp(-jkx) + S_2(\omega)\exp(+jkx)$

$$p(x,t) = s_1(t - \frac{x}{c}) + s_2(t + \frac{x}{c})$$

How to make a (n almost) plane wave?

- Flat source much larger than wavelength (cf. loudspeaker cone at high frequencies)
- Wave tube with cross section dimensions smaller than half a wavelength (cf. bass reflex port, wind instrument)
- Note: each wave field can be *decomposed* into plane waves by spatial Fourier Transform! (beyond the scope of this presentation)

Diffuse field

- A diffuse (= homogeneous and isotropic) field consists of an *infinite* number of *equally strong*, *uncorrelated* plane waves, equally distributed over *all directions* of incidence
- → does not exist in practice
- Nevertheless: a sound field in a room/hall is often considered as diffuse.....

Properties of plane waves

- They do not decrease in amplitude when propagating (we 'forgot' the damping in the derivation of our wave equation!)
- $p = \rho c v$: pressure and particle velocity are in phase and differ a constant real factor (the wave impedance) in amplitude
- In air at 20 deg. C: $\rho c \approx 400$

3D reality:

p is (still) scalar, $\vec{v}$ is now vector;

$$\frac{\partial}{\partial x} \to \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \nabla; \ \nabla p \text{ is gradient (vector),}$$

$\nabla . \vec{v}$ is divergence (scalar);

$$\frac{\partial^2}{\partial x^2} \to \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2$$

3D wave -, Helmholtz equation

3D wave equation:

$$\nabla^2 p(x,y,z,t) - \frac{1}{c^2}\frac{\partial^2 p(x,y,z,t)}{\partial t^2} = 0$$

3D wave -, Helmholtz equation

3D wave equation:

$$\nabla^2 p(x,y,z,t) - \frac{1}{c^2}\frac{\partial^2 p(x,y,z,t)}{\partial t^2} = 0;$$

3D Helmholtz equation:

$$\nabla^2 P(x,y,z,\omega) + k^2 P(x,y,z,\omega) = 0$$

3D wave -, Helmholtz equation

3D wave equation:

$$\nabla^2 p(x,y,z,t) - \frac{1}{c^2}\frac{\partial^2 p(x,y,z,t)}{\partial t^2} = 0;$$

3D Helmholtz equation:

$$\nabla^2 P(x,y,z,\omega) + k^2 P(x,y,z,\omega) = 0;$$

there is an infinite number of spatial solutions

Oplossen op begin- en randvoorwaarden

Wave equation in spherical coordinates (1)

- Relevant if there is some form of rotation symmetry
- Coordinates: radius *r*, elevation angle θ, azimuth angle ϕ:

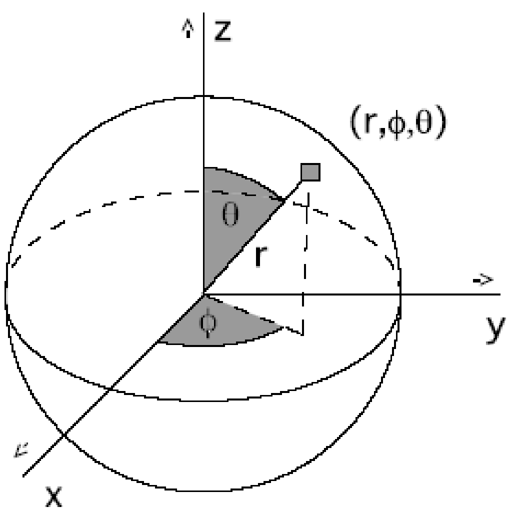


Abbildung: Definition der Kugelkoordinaten

Die Umrechnung geschieht nach:

$$r = \sqrt{x^2 + y^2 + z^2},$$
$$\theta = \tan^{-1}[\sqrt{x^2 + y^2}/z],$$
$$\phi = \tan^{-1}[y/x].$$

Wave equation in spherical coordinates (2)

- Result of ‘Umrechnung’ and FT:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial P}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial P}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 P}{\partial\phi^2}+kP^2=0$$

Monopole source (1)

- Only *r*-dependency → Helmholtz equation:

$$\frac{\partial^2 P}{\partial r^2} + kP^2 = 0$$

Monopole source (2)

The solution of the 3D wave equation for an omnidirectional point source (monopole source) is spherical:

$$p(r,t) = s(\frac{t - \frac{r}{c}}{r});$$

In the space-frequency domain:

$$P(r,\omega) = S(\omega)\frac{\exp(-jkr)}{r}$$

How to make a (n almost) spherical wave

- ‘Breathing sphere’ (not very practical)
- Piston (small in relation to wavelength) in closed box (cf. loudspeaker at low frequencies)

Properties of spherical waves

- They are ‘spatially neutral’
- Their amplitude decreases with $1/r$:
  “-6 dB per distance doubling”
- For large r, they approximate plane waves

Cylindrical solution

* cylindrial solution (monopole line source):

$$p(r,t) = s\left(\frac{t-\frac{r}{c}}{\sqrt{r}}\right); \quad P(r,\omega) = S(\omega)\frac{\exp(-jkr)}{\sqrt{r}}$$

* 3 dB decrease per distance doubling
* m.l. sources:
  - road with traffic stream
  - loudspeaker column

Wave equation in spherical coordinates (2)

- Result of 'Umrechnung' and FT:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial P}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial P}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 P}{\partial\phi^2}+kP^2=0$$

- General solution by separation of variables:

$$P(r,\theta,\phi,\omega)=R(r)\Theta(\theta)\Phi(\phi)\Omega(\omega)$$

Plane wave decomposition

- Each sound field can be decomposed into *plane waves* by *Spatial Fourier Transform*

what again was the motivation? →

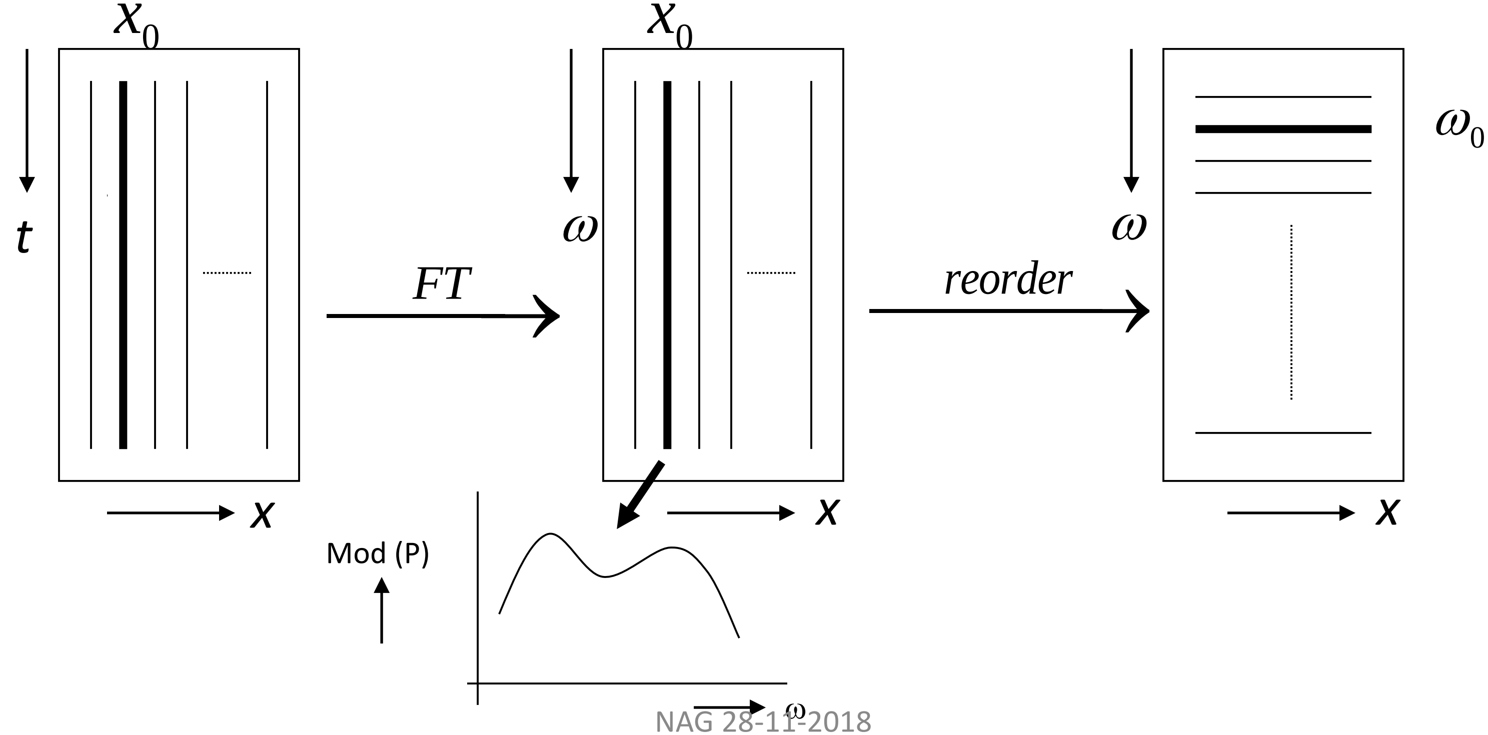
- Plane waves can easily be extrapolated (only phase shift)
- This way, *measurements at other array positions can be simulated* → one array measurement gives data for complete hall!

Suppose:

- We did a multi-trace sound pressure measurement $p(x, y, z_0, t)$ in the horizontal plane $z = z_0$
- For simplicity: 1D array → *y* = 0; leave *y* out
- Most processing in frequency domain → FT from t to f, ω: $p(x, z_0, t) \xrightarrow{FT} P(x, z_0, \omega)$
- This means: for each *x*, the time signal is decomposed into a number of harmonic (sinusoidal) functions ('tones') with different frequencies, each having a specific amplitude and phase: for $x = x_0$, we have a (complex) frequency spectrum $P(x_0, z_0, \omega)$

Reordering the data

- *Reorder* the data such that for each frequency value $\omega = \omega_0$ we have a (complex) space function $P(x, z_0, \omega_0)$
- The process until now in figure:



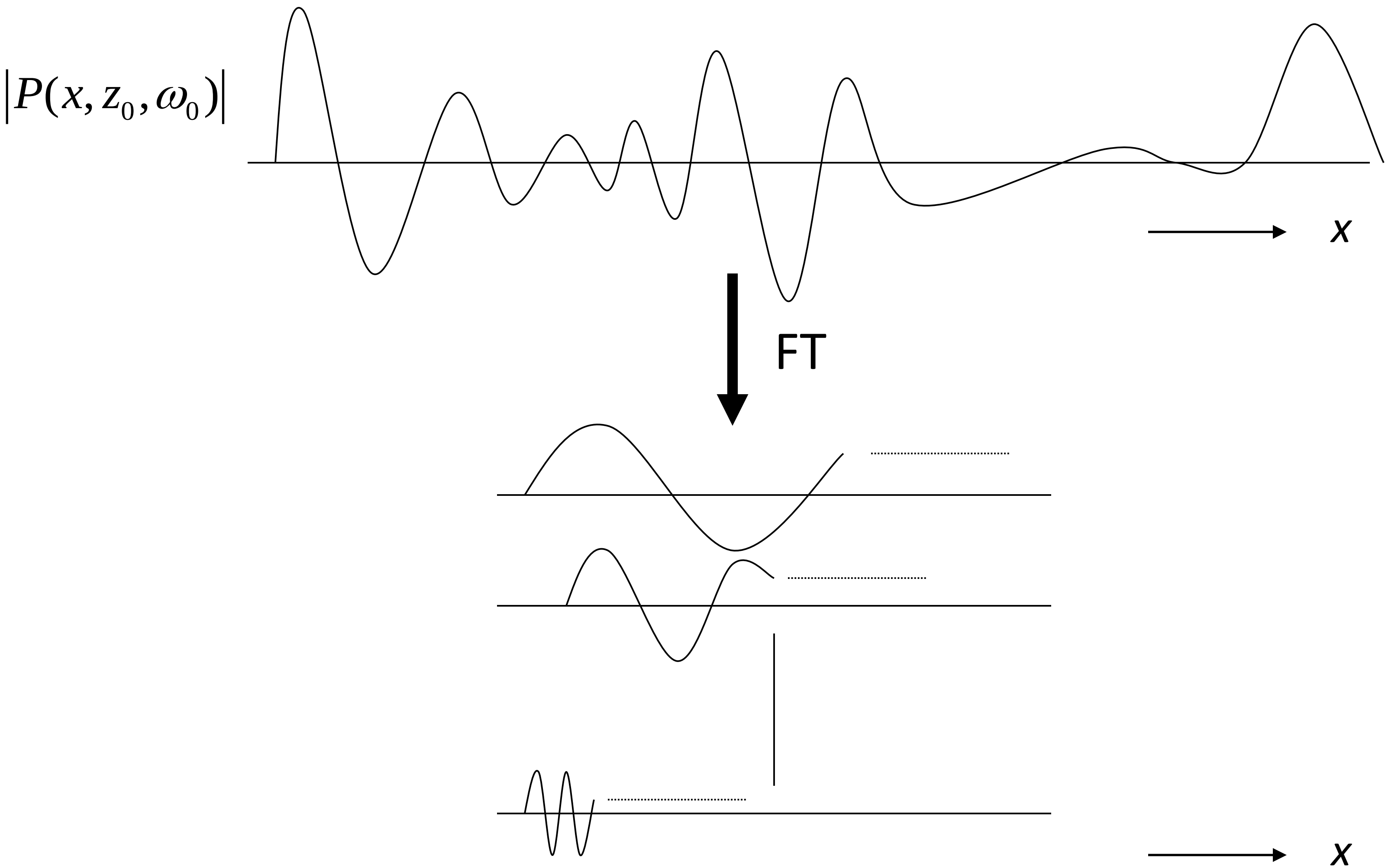
$P(x, z_0, \omega_0)$ to play with…

- It represents the (complex) pressure distribution along *x*, due to a 'source' generating a 'tone' with one single frequency $f_0 = \omega_0 / 2\pi$ corresponding with one wavelength λ_0
- Let's submit $P(x, z_0, \omega_0)$ to a FT operation:
- The (stupid) computer decomposes $P(x, z_0, \omega_0)$ into 'sinuses' with different 'frequencies' each having a specific amplitude and phase:

$$P(x, z_0, \omega_0) \xrightarrow{FT} \tilde{P}(?, z_0, \omega_0)$$

- Let's call the new variable k_x for the moment

In figure:



Nonsens, or …. physics? (1)

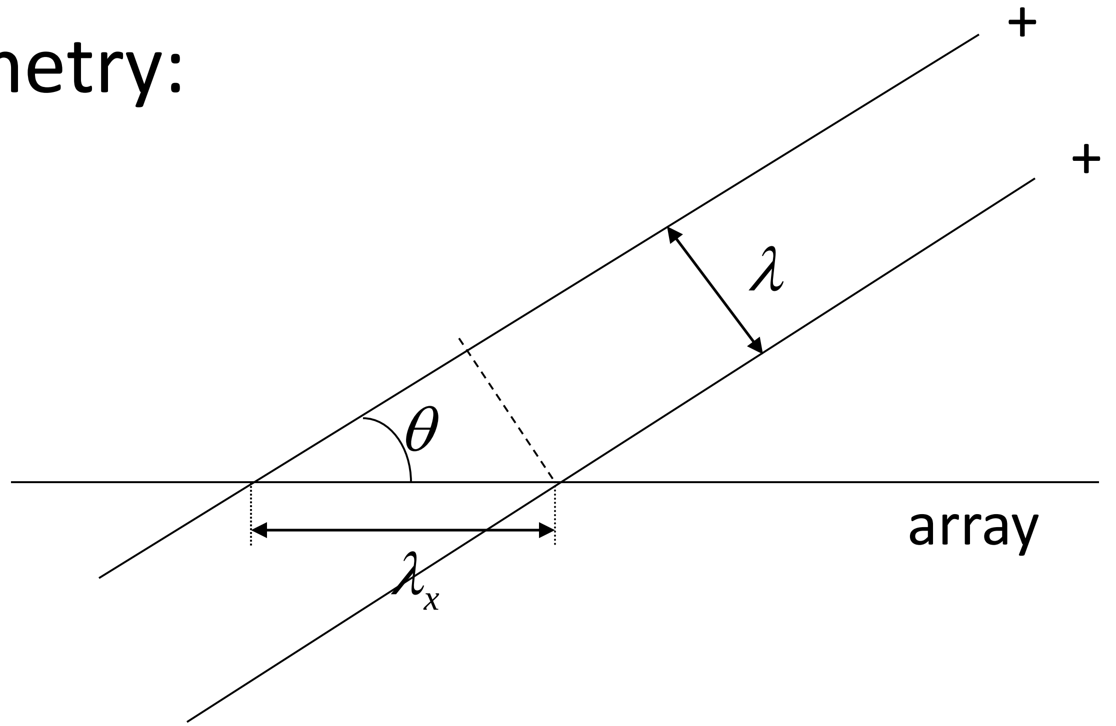
- Note: a harmonic 'tone' is also harmonic (= sinusoidal) in *space*: it has a *wavelength* λ as an equivalent of the temporal *period T*
- So, it makes sense to relate the different 'frequencies' into which a *spatial* data set is decomposed by FT to wavelength λ
- Analogy: in the temporal domain we go from t to $\omega = 2\pi / T$, so let's go in the spatial domain from x to 'spatial frequency' $k_x = 2\pi / \lambda_x$ where λ_x is the "apparent wavelength"

Nonsens, or …. physics? (2)

- Question: how can we have different wavelengths λ_x along the *x*-axis (for each 'sp.fr.' another one) for a 'pure tone' with one frequency $f_0 = \omega_0 / 2\pi$ and thus one wavelength λ_0 ??
- Answer: harmonic *plane waves* incident on plane $z = z_0$ under different *angles* θ create different 'apparent wavelengths' λ_x along the *x*-axis

Apparent wavelength

- Geometry:



- In formula: $\lambda_x = \dfrac{\lambda}{\sin\theta}$

Nonsens, or …. physics? (2)

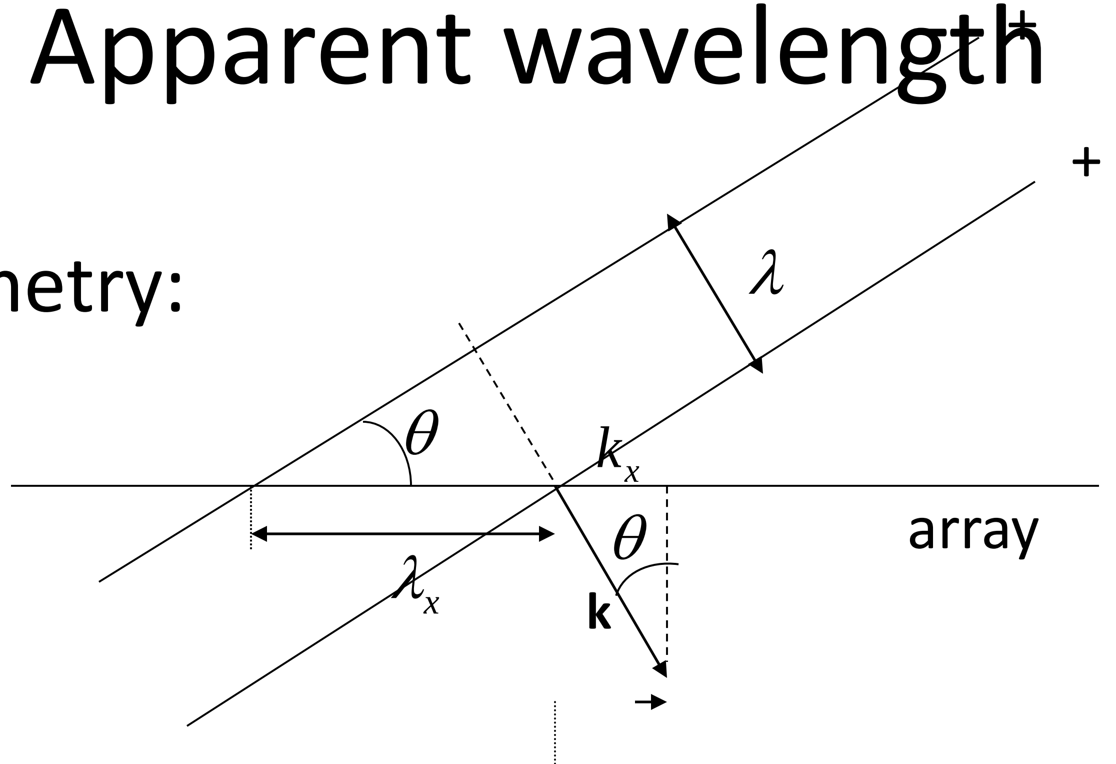
- Question: how can we have different wavelengths λ_x along the *x*-axis (for each 'sp.fr.' another one) for a 'pure tone' with one frequency $f_0 = \omega_0 / 2\pi$ and thus one wavelength λ_0 ??
- Answer: harmonic *plane waves* incident on plane $z = z_0$ under different *angles* θ create different 'apparent wavelengths' λ_x along the *x*-axis
- So: spatial FT decomposes (for each ω_0) the wavefield along *x* into (the result of) incident plane waves with different 'spatial frequencies' k_x

Spatial frequency and incidence angle

- What does k_x have to do with θ?
- Look at the figure again, and derive some more formulas

Apparent wavelength

- Geometry:



$$\lambda_x = \frac{\lambda}{\sin\theta} \;,\; c_x = \frac{c}{\sin\theta}$$

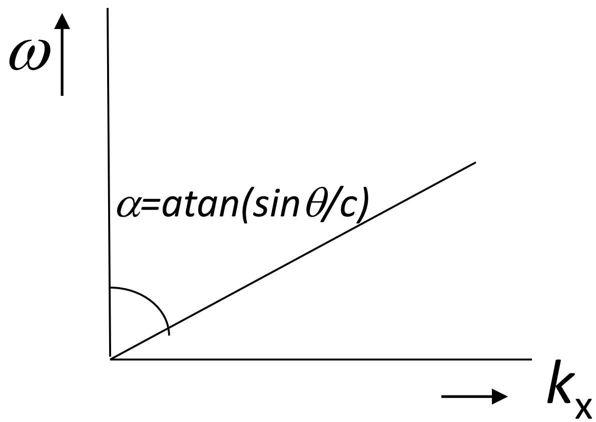
- In formula: $k = |\mathbf{k}| = \frac{\omega}{c} \xrightarrow{define} k_x = \frac{\omega}{c_x} = k\sin\theta$

Spatial frequency and incidence angle

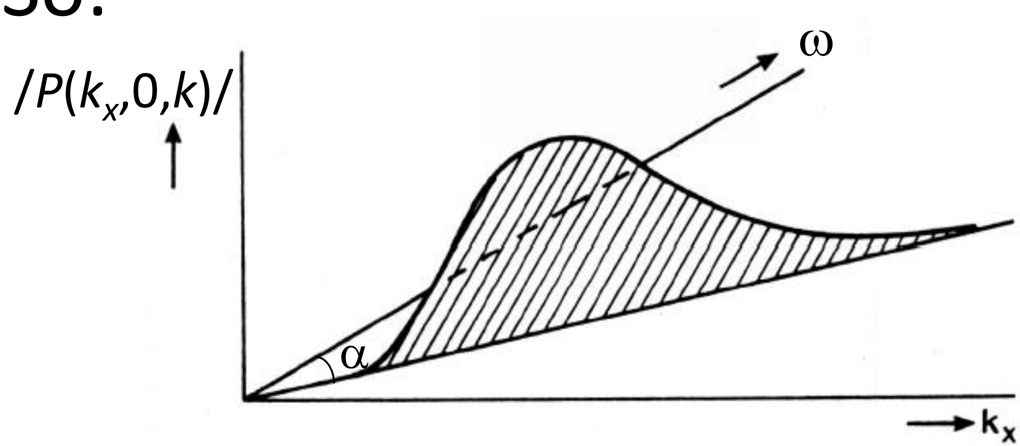
- What does k_x have to do with θ?
- Look at the figure again, and derive some more formulas
- So, 'spatial frequency' k_x is the *x*-component of the wave vector **k** along the *x-axis,* specifying the incidence angle θ
- We call the (k_x, ω)- domain the 'wave number - frequency' domain
- Sometimes given as (k_x,f) – scaling of horizontal axis with 2π

Plane wave in (k_x, ω)-domain

- Angle of incidence with *x*-axis ('array'): θ
- All data on line $k_x = k\sin\theta = \dfrac{\omega}{c}\sin\theta$:

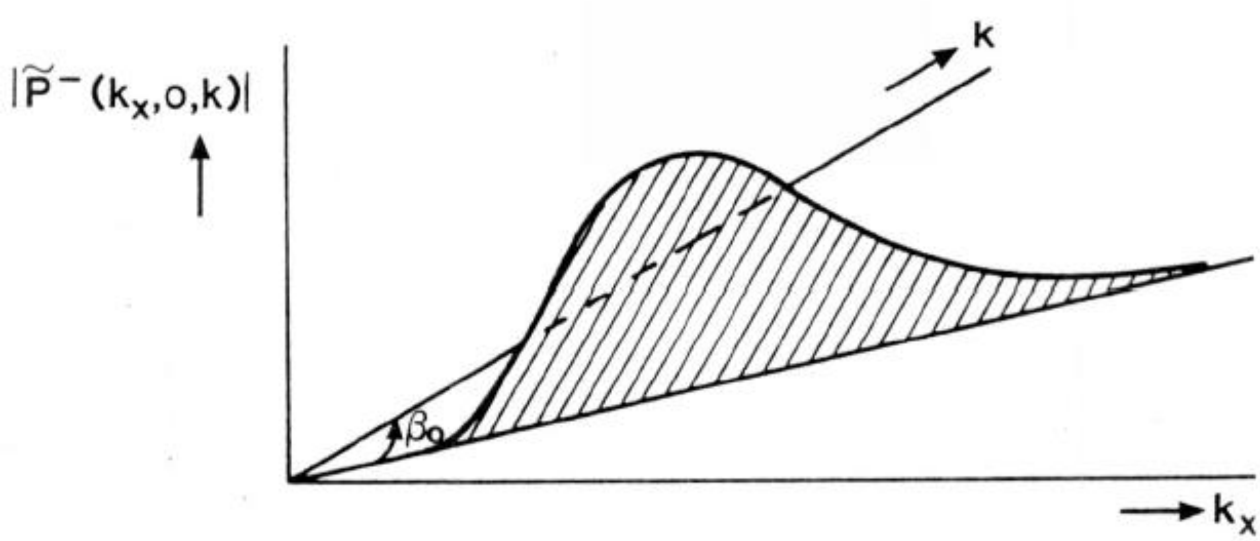


- So:



Plane wave in (k_x, k)-domain

- Give both axes same dimensions
- Then:

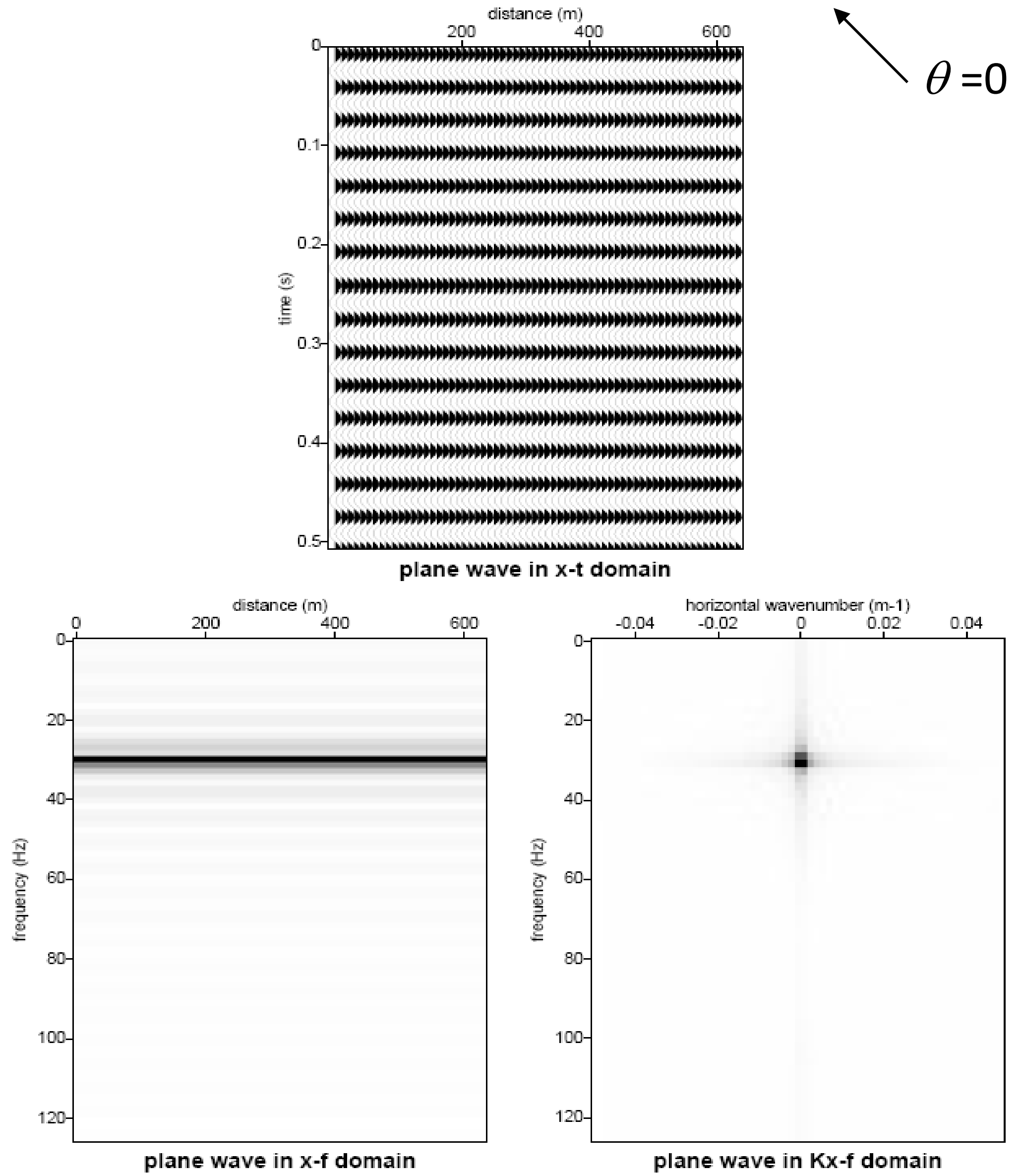


- With: β_0=arctan(sinθ)

Some simulated examples

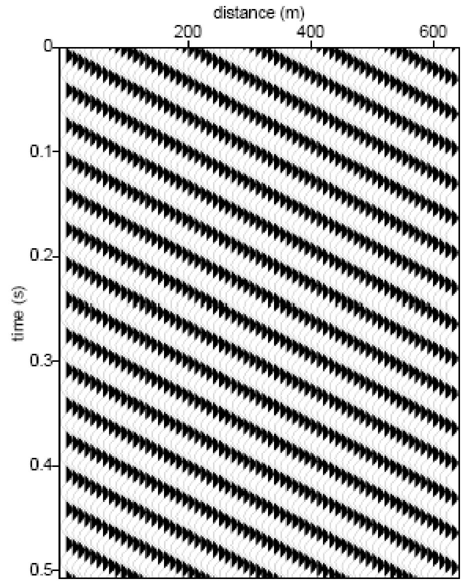
- By courtesy of my seismic imaging colleague Dr Eric Verschuur

horizontal plane wave; f=30 Hz, p=0

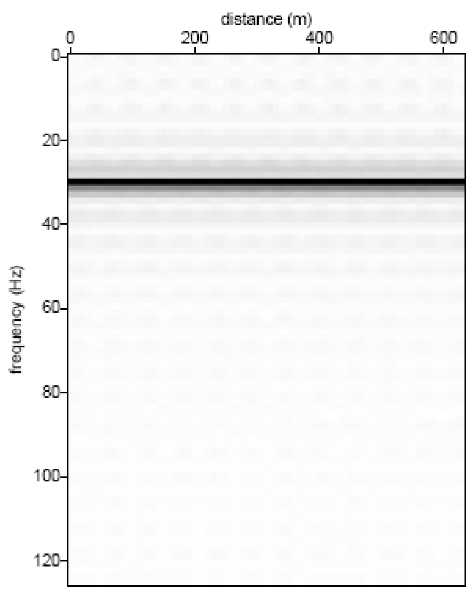


dipping plane wave; f=30 Hz, p=0.3e-3

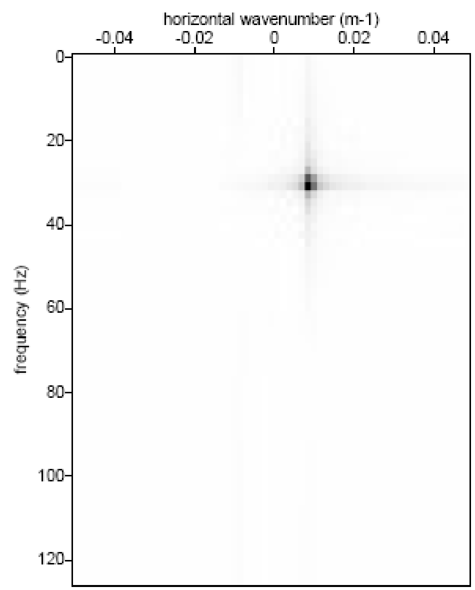
$p=\sin\theta/c$: ‘slowness’



plane wave in x-t domain

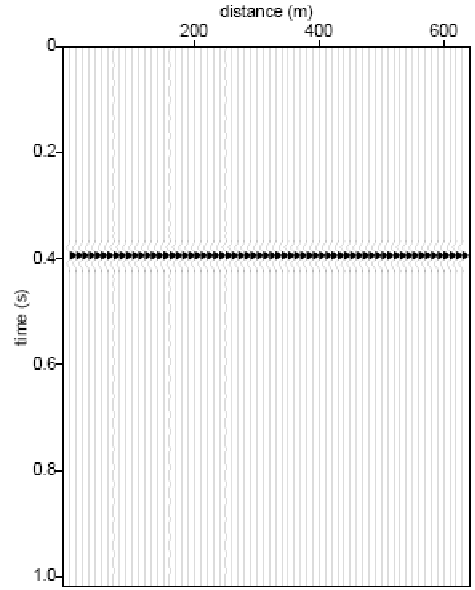


plane wave in x-f domain

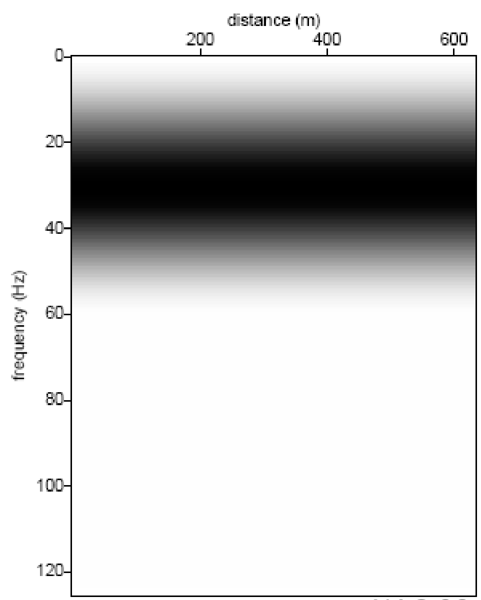


plane wave in Kx-f domain

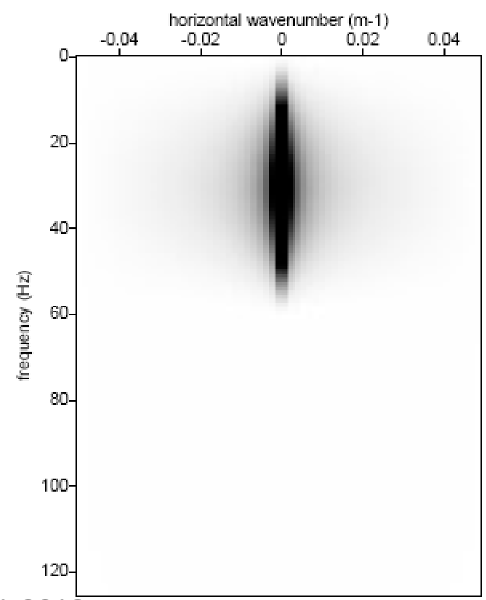
horizontal plane wave (broadband)



plane wave in x-t domain

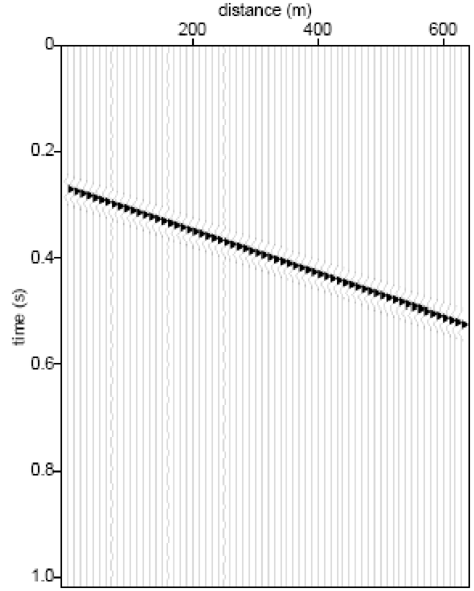


plane wave in x-f domain

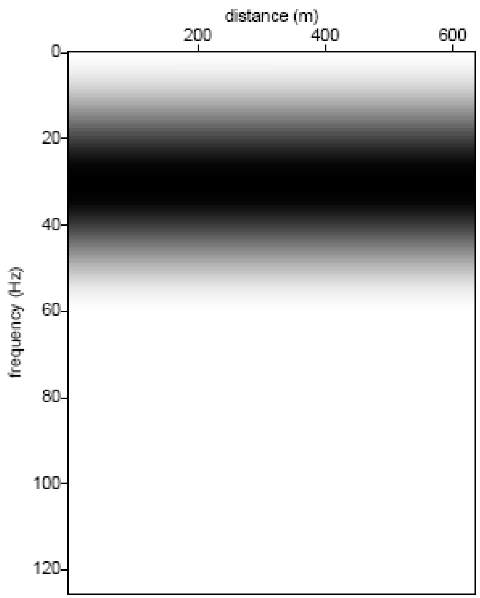


plane wave in Kx-f domain

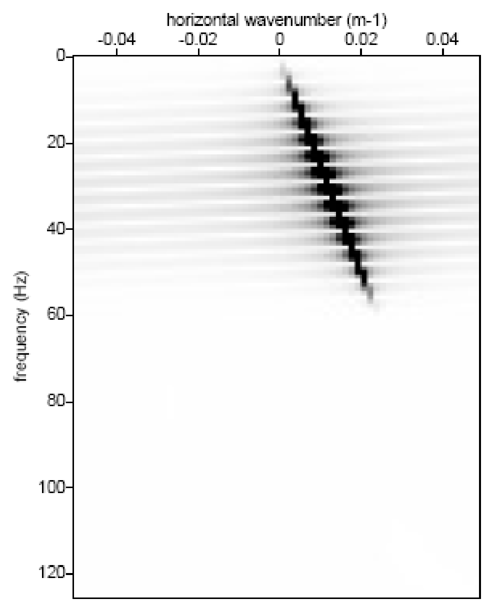
dipping plane wave (broadband)



plane wave in x-t domain

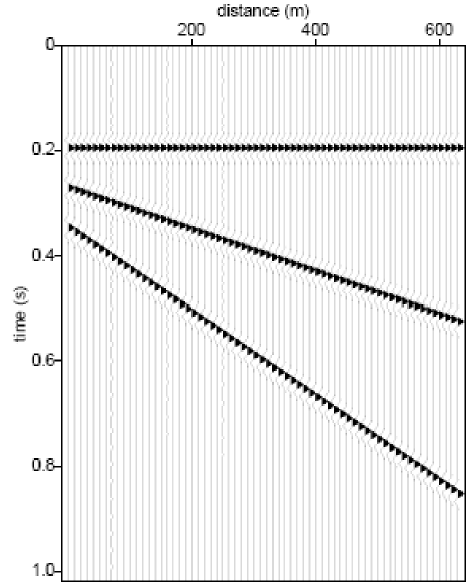


plane wave in x-f domain

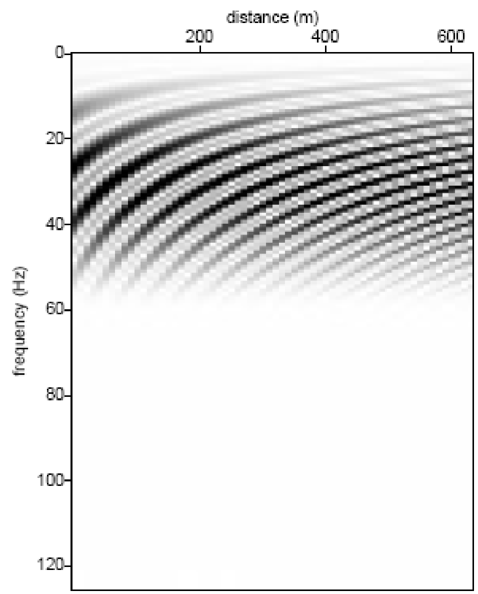


plane wave in Kx-f domain

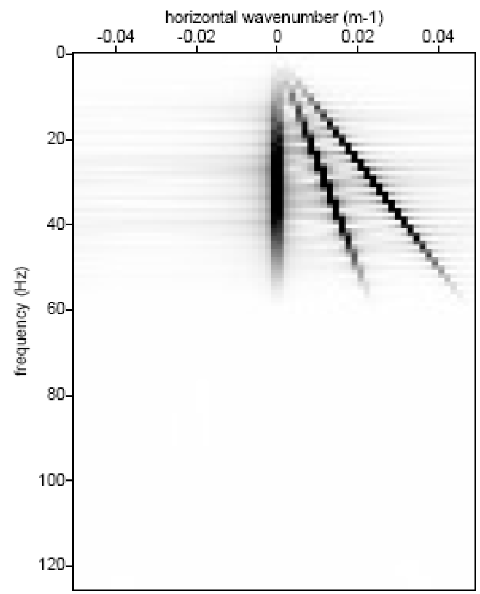
3 plane wave sections



3 plane waves in x-t domain

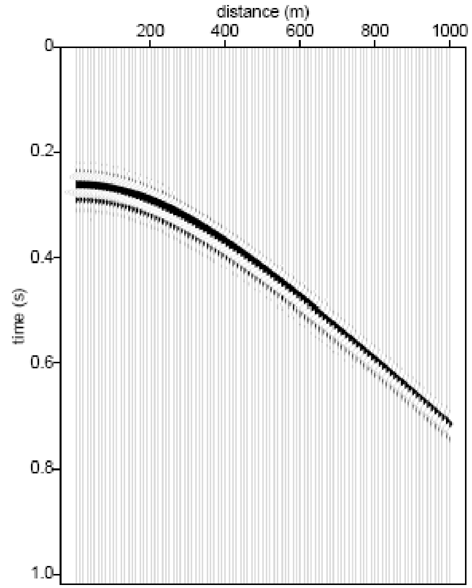


3 plane waves in x-f domain

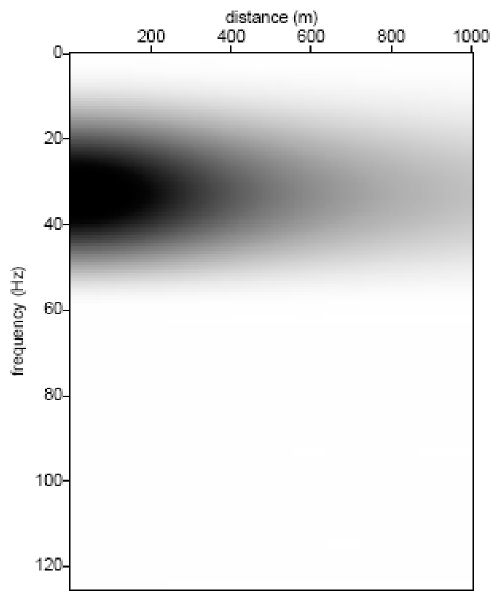


3 plane waves in Kx-f domain

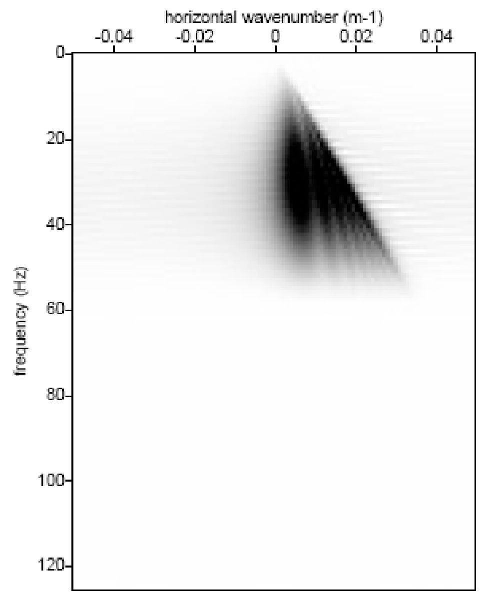
reflection of 1 layer



reflection in x-t domain



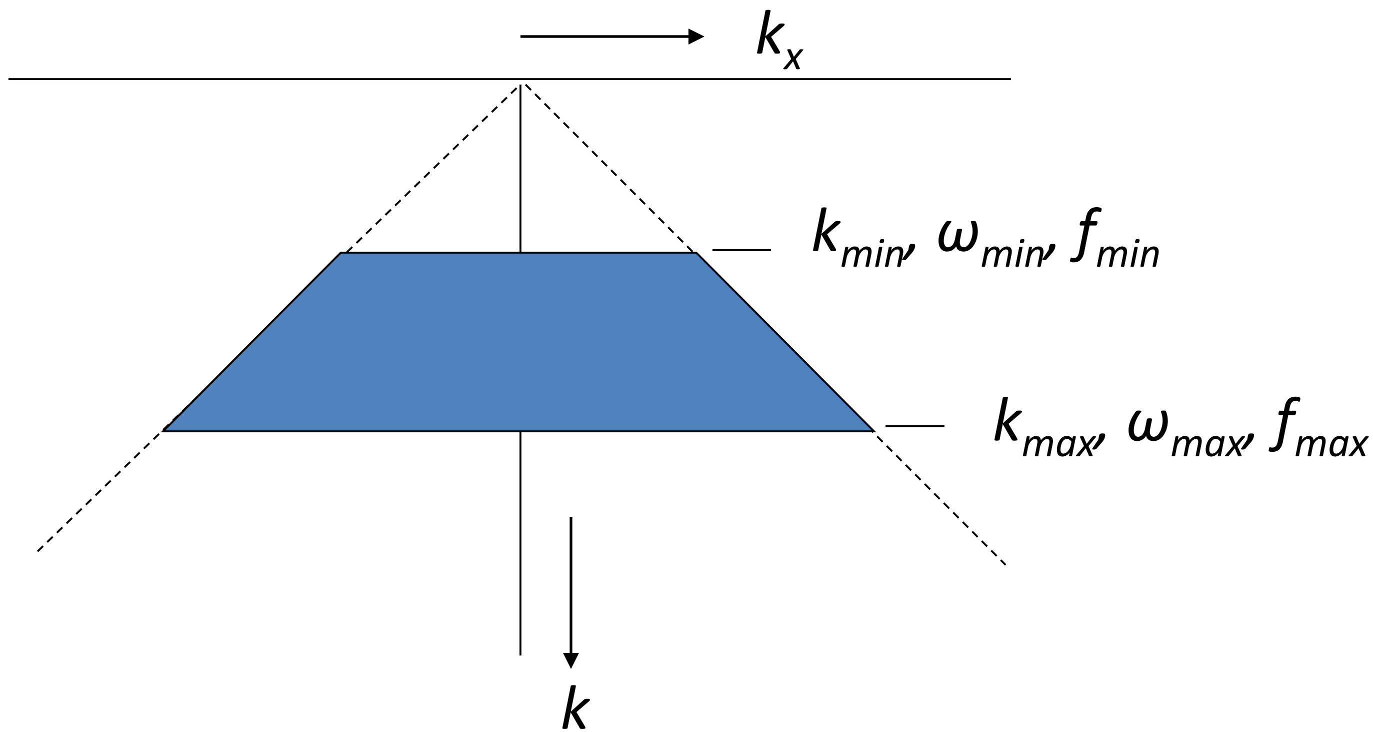
reflection in x-f domain



reflection in Kx-f domain

Recording of full orchestra with a microphone array

- Schematically:

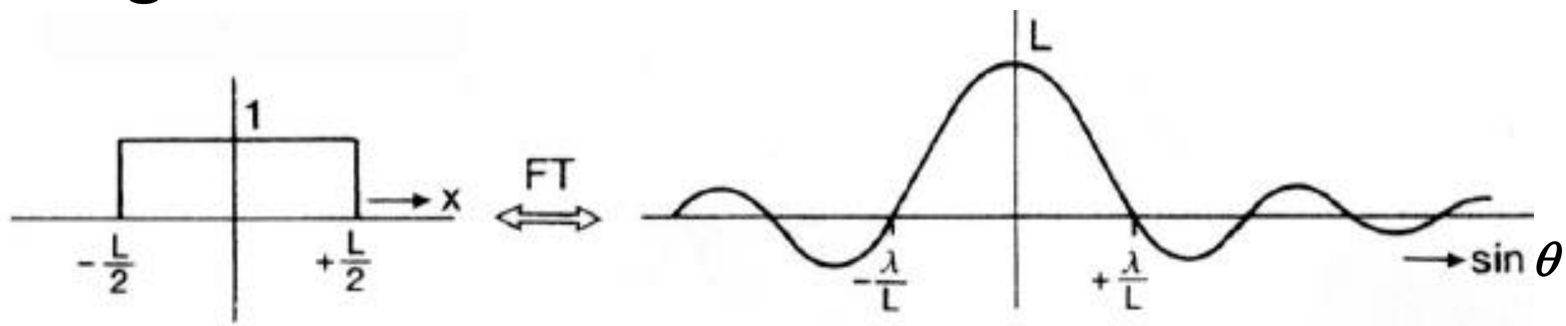


Spatial FT and directivity

- Spatial FT: decomposition of a spatial function into plane waves from different directions
- Directivity pattern: indicates sensitivity of a(n array of) transducers for plane waves in different directions
- Hence: spatial FT of transducer configuration yields its *directivity characteristics*

Famous FT: box → sinc

- In figure:



- In formula:

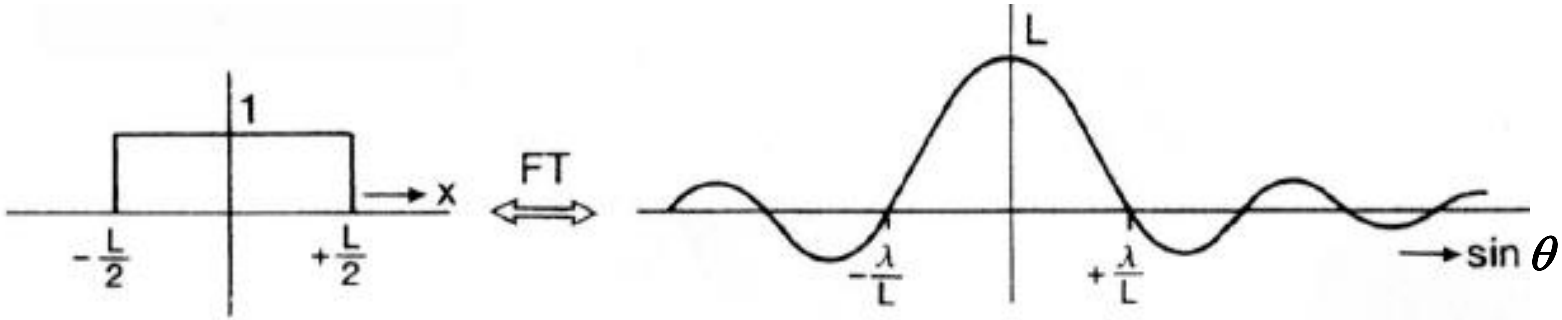
$$A(x,\omega) = 1 \text{ for } -L/2 \le x \le L/2$$

$$A(x,\omega) = 0 \text{ elsewhere} \underset{\longleftarrow}{\overset{\text{FT}}{\longrightarrow}}$$

$$\tilde{A}(k_x,\omega) = \frac{\sin(k_x L/2)}{k_x/2};\ k_x = \frac{2\pi}{\lambda}\sin\theta$$

Box and linear array

- Figure again:



- The box can be seen as a continuous array of equally driven loudspeakers or microphones, and the spatial FT as its directivity pattern: a main lobe (‘beam’) and small side lobes
- The main lobe width is determined by λ/L: a long (*re* wavelength) array (e.g. loudspeaker column) yields a narrow beam